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A note on the $(1, 1, \ldots, 1)$ monopole metric

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Abstract

Recently Lee et al. in CU-TP-739, hep-th/9602167, calculated the asymptotic metric on the moduli space of $(1, 1, \ldots, 1)$ BPS monopoles and conjectured that it was globally exact. In this paper it is shown that this conjecture is true for the corresponding moduli space of Nahm data.

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1. Introduction

For some time now there has been considerable interest in the natural hyperkaehler metric on the moduli space of $SU(2)$ monopoles in $\mathbb{R}^3$. It is known from the work of Taubes that a monopole near the boundary of the moduli space of monopoles of charge m approximates a collection of m charge one monopoles. It was argued by Manton [16] that the geodesics of this metric correspond to scattering of m slowly moving monopoles. There are now many interesting examples of scattering of $SU(2)$ monopoles beginning with the calculation of the metric on the moduli space of $SU(2)$ charge two monopoles by Atiyah and Hitchin [1] and more recently results on the scattering of monopoles with special symmetry [8–12].

Monopoles also exist for compact groups G other than $SU(2)$. We will be interested only in the case of maximal symmetry breaking. In this case the particles making up the monopole come in r distinguishable 'types' where r is the rank of the group G. The r types correspond to the r different elementary ways of embedding $SU(2)$ into G along simple root directions. The magnetic charge of a G monopole is a vector $m = (m_1, \ldots, m_r)$ where

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m_i can be thought of as the number of monopoles of type i [17]. If any of the m_i vanish, the monopole is obtained from an embedded subgroup; so the simplest monopole that is genuinely a monopole for G is one with each $m_i = 1$. We are interested in the structure of the moduli space for this case and its metric. Note that in general the moduli space has dimension $4(m_1 + \ldots + m_r)$ so that the moduli space of $(1, 1, \ldots, 1)$ monopoles has dimension $4r$.

The existence of a hyperkaehler metric on the monopole moduli space is suggested by the fact that it can be realised as an infinite-dimensional hyperkaehler quotient. In the case of $SU(2)$ Atiyah and Hitchin used analytic results of Taubes to rigourously establish that such a metric did indeed exist. In the case of other gauge groups and maximal symmetry breaking these results have not been established although they are believed to be true. Most work on the metric on the moduli space of monopoles for higher rank gauge groups, instead of studying the moduli space directly, has worked with the space of Nahm data. It is known [13,19] that the space of Nahm data is diffeomorphic to the monopole moduli space for $SU(n)$ monopoles with maximal symmetry breaking. The space of Nahm data can also be realised as an infinite-dimensional hyperkaehler quotient and hence one expects it to also have a hyperkaehler metric. Moreover, it is conjectured that this metric is the same as the monopole metric. For $SU(2)$ this has been proved by Nakajima [18]. We will see below that working with the Nahm data has the advantage that for simple cases such as $(1, 1, \ldots, 1)$ monopoles we can realise it as a *finite-dimensional* hyperkaehler quotient and a hyperkaehler metric can be rigourously and explicitly constructed. It has the disadvantage, of course, that without a generalisation of Nakajima's result, we do not know that the metric we have constructed is the monopole metric.

For the group $SU(3)$ the rank is two and the metric on the moduli space of $(1, 1)$ monopoles was studied by Connell [3,4]. The same result was also obtained independently by Gauntlett and Low [5] and Lee et al. [5,14]. In these latter works some special assumptions on the values of Higg's field at infinity that simplified the work of Connell are removed. The metric obtained is globally of Taub-NUT type.

For the more general case of an $SU(n+1)$ monopole of charge $(1, \ldots, 1)$ Lee et al. [15] calculate the asymptotic form of the monopole metric and show that it is asymptotically Taub-NUT. They then give an argument that the asymptotic form of the metric can be smoothly extended to the whole moduli space and they conjecture that the monopole metric is indeed exactly this extended metric. In this note I will show that this conjecture is true for the metric on the Nahm data of $SU(n+1)$, $(1, 1, \ldots, 1)$ monopoles.

After completing this note a preprint appeared in Chalmers [2] that proves more directly that the $(1, \ldots, 1)$ monopole moduli space metric is of the type conjectured in [15].

In summary the paper is as follows: Section 2 reviews the hyperkaehler quotient construction applied to quaternionic vector spaces. Section 3 describes the infinite-dimensional hyperkaehler quotient that defines $\mathcal{N}$ the moduli space of $(1, \ldots, 1)$ Nahm data and shows that it can be realised as a finite-dimensional hyperkaehler quotient. This enables a rigourous definition of the metric on $\mathcal{N}$ as a hyperkaehler quotient of a finite-dimensional hyperkaehler manifold. This is described in Section 4 and in Section 5 it is shown that the moduli space is isometric to a product

$$\mathcal{N} = \mathbb{R}^3 \times \frac{\mathcal{N}_c \times \mathbb{R}}{\mathbb{Z}},$$

where $\mathcal{N}_c$ is the space of Nahm data corresponding to centred monopoles, and $\mathbb{R}^3$ and $\mathbb{R}$ are given a multiple of the standard metric. Finally in Section 6 we consider the metric on $\mathcal{N}_c$. The space $\mathcal{N}_c$ is just $\mathbb{H}^{n-2}$ where $\mathbb{H} = \mathbb{R}^4$ is quaternionic space. In the case of $SU(3)$ it is possible to give an explicit formula for the metric on this space [3,4], in the present case I use a result of Hitchin [6] to show that it has the same form as the metric in [15].

2. Hyperkaehler quotients of vector spaces

A hyperkaehler manifold [6] is a Riemannian manifold (M, g) with three complex structures I, J and K which satisfy the quaternion algebra and are covariantly constant.

We need to consider from [7] the hyperkaehler quotient of a hyperkaehler manifold by a group. For our purposes it is enough to consider the case when the manifold in question is a vector space. Let V be a real vector space with three complex structures $e_1 = I$, $e_2 = J$, $e_3 = K$ which satisfy the quaternion algebra. Assume also that V has an inner product $\langle\ ,\ \rangle$ which is preserved by each of the e_i. Then V has three symplectic forms ω_i defined by $\omega_i(v, w) = \langle v, e_i w\rangle$. Since the tangent space at any point of V is canonically identified with V itself this makes V a hyperkaehler manifold.

Assume now that a group G acts freely on V in such a way that V/G is a manifold and $V \to V/G$ is a principal G fibration. Assume further that the G action preserves the inner product on the tangent spaces of V and also commutes with the action of the e_i. If ξ is an element of LG, the Lie algebra of G, it defines a vector field $\iota(\xi)$ on V. The *moment map*

$$\mu : V \to \mathbb{R}^3 \otimes LG^*$$

of this group action is defined as follows. For any $\xi \in LG$ and $v \in V$ we define $\mu(v)(\xi) = (\mu_1(v)(\xi), \mu_2(v)(\xi), \mu_3(v)(\xi))$ in $\mathbb{R}^3$ by

$$\mu_k(v)(\xi) = \int_0^1 \omega_k(\iota(\xi)(tv), v)\,\mathrm{d}t = \int_0^1 \langle \iota(\xi)(tv), e_k(v)\rangle\,\mathrm{d}t \tag{2.1}$$

for each $k = 1, 2, 3$.

Let $V_0 = \mu^{-1}(0)$, then the hyperkaehler quotient of V is the space V_0/G. To see that this is a hyperkaehler manifold let π be the projection $V_0 \to V_0/G$. If $x \in V_0/G$, choose $\hat{x} \in \pi^{-1}(x) \subset V_0$. We can split the tangent space at $\hat{x} \in V_0$ into vertical directions tangent to the G action and horizontal directions which are orthogonal to the vertical directions. The horizontal directions are naturally identified with the tangent space to V_0/G at x and this enables us to define an inner product and a hyperkaehler structure on that tangent space. This construction is, in fact, independent of the choice of $\hat{x}$ in $\pi^{-1}(\pi(x))$ because of the G invariance. I refer the reader to [7] for details.

3. Moduli space of Nahm data

We are interested in $SU(n+1)$ monopoles or more precisely their Nahm data. In the interests of brevity I will not review the theory of monopoles or the relationship between monopoles and solutions of Nahm's equations but refer the reader to [1] and references therein for the $SU(2)$ theory and to [13,17,19] for the $SU(n+1)$ theory. Higg's field at infinity of the monopole has eigenvalues $\mathrm{i}\lambda_0, \dots, \mathrm{i}\lambda_n$ where we assume that

$$\lambda_0 < \lambda_1 < \cdots < \lambda_{n-1} < \lambda_n.$$

The fact that these eigenvalues are distinct is called maximal symmetry breaking at infinity.

Denote by $\tilde{\mathcal{A}}$ the set of all pairs (T, a) where $T = (T^1, \dots, T^n)$ and each

$$T^i : [\lambda_{i-1}, \lambda_i] \to \mathbb{H}$$

is a smooth function, and $a = (a^1, \dots, a^{n-1}) \in \mathbb{H}^{n-1}$. It is useful to think of the vector $a = (a^1, \dots, a^{n-1})$ as a function on the set $(\lambda_1, \dots, \lambda_{n-1})$ whose value at λ_i is just a^i. We will consider the space $\tilde{\mathcal{A}}$ as a left quaternionic vector space.

Denote by $\tilde{\mathcal{G}}$ the set of all $g = (g^1, \dots, g^n)$ where each g^i is a smooth map

$$g^i : [\lambda_{i-1}, \lambda_i] \to U(1)$$

and

$$1 = g^1(\lambda_0), \dots, g^i(\lambda_i) = g^{i+1}(\lambda_i), \dots, g^n(\lambda_n) = 1. \tag{3.1}$$

The set $\tilde{\mathcal{G}}$ forms a group under pointwise multiplication.

The group $\tilde{\mathcal{G}}$ acts on the right of $\tilde{\mathcal{A}}$ by

$$(gT)^j = T^j + \frac{1}{\mathrm{i}}\frac{\mathrm{d}g^j}{g^j}, \qquad (ga)^j = a^j g^j(\lambda^j) = a^j g^{j+1}(\lambda^j).$$

We define an inner product on $\tilde{\mathcal{A}}$ by

$$\langle (T, a), (S, b)\rangle = \sum_{i=1}^{n} \int_{\lambda_{i-1}}^{\lambda_i} \mathrm{Re}(T^i \bar{S}^i) + \sum_{i=1}^{n-1} \mathrm{Re}(a^i \bar{b}^i).$$

This inner product makes $\tilde{\mathcal{A}}$ an (infinite-dimensional) hyperkaehler vector space. We want to consider its hyperkaehler quotient. It is easy to check that the group action preserves the hyperkaehler structure. It is not clear, because of the infinite dimensionality, that the quotient is nicely behaved. In the next section we shall see that we can avoid this problem by replacing $\tilde{\mathcal{A}}$ by a finite-dimensional vector space and forming the hyperkaehler quotient of that instead. To motivate the choice of finite-dimensional space let us continue and calculate the formal infinite-dimensional hyperkaehler quotient of $\tilde{\mathcal{A}}$.

To define the moment maps for the action of $\tilde{\mathcal{G}}$ we need to consider the infinitesimal action of the Lie algebra $L\tilde{\mathcal{G}}$. The Lie algebra $L\tilde{\mathcal{G}}$ is the set of all $\xi = (\xi^1, \dots, \xi^n)$ where each

$$\xi^i : [\lambda_{i-1}, \lambda_i] \to \mathbb{R}$$

is a smooth map. By differentiating 3.1 we see that ξ has to satisfy

$$0 = \xi^1(\lambda_0), \dots, \xi^i(\lambda_i) = \xi^{i+1}(\lambda_i), \dots, \xi^n(\lambda_n) = 0.$$

Of course the image of ξ should be in the Lie algebra of $U(1)$ but we have identified that with $\mathbb{R}$. We fix our conventions for that identification by noting that the exponential map for the group $\tilde{\mathcal{G}}$ is $\xi \mapsto \exp(2\pi \mathrm{i}\xi)$.

An element $\xi \in L\tilde{\mathcal{G}}$ then defines a vector field $\iota(\xi)$ on $\mathcal{A}$ whose value at (T, a) is

$$\iota(\xi)(T, a) = ((2\pi\, \mathrm{d}\xi^1, \dots, 2\pi\, \mathrm{d}\xi^n), (2\pi a^1 \mathrm{i}\xi^1(\lambda_1), \dots, 2\pi a^{n-1}\mathrm{i}\xi^{n-1}(\lambda_{n-1})).$$

We can now calculate the moment map from 2.1 and we find that (T, a) is in the kernel of μ if and only if

$$\mathrm{Re}(\mathrm{d}T^j) = 0$$

for each $j = 1, \dots, n$ and

$$\mathrm{Im}(T^{j+1} - T^j) = \tfrac{1}{2} a^j \mathrm{i} \bar{a}^j$$

for each $j = 1, \dots, n-1$.

It is clear from these equations that to describe the hyperkaehler quotient of $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{G}}$ we could restrict our attention from $\tilde{\mathcal{A}}$ to the subset of pairs (T, a) where the imaginary part of T is constant. If we do that and wish to still have a hyperkaehler structure then we will need to restrict attention to T whose real part is also constant. Notice that if we start out with a T which is real then by integrating starting at λ_0 we can construct a $g = (g^1, \dots, g^n)$ such that $gT = 0$ and satisfying all the conditions to be in $\tilde{\mathcal{G}}$ except that we may not have $g(\lambda_n) = 1$. But in that case we can find an h such that $\mathrm{d}h$ is constant and $h(\lambda_n) = g(\lambda_n)^{-1}$. The product gh is in $\tilde{\mathcal{G}}$ and gT has constant real part. We conclude that every (T, a) in $\mu^{-1}(0)$ can be gauge transformed so that gT is constant.

4. Hyperkaehler quotient

Denote by $\mathcal{A}$ the set of all pairs (τ, a) where $\tau \in \mathbb{H}^n$ and $a \in \mathbb{H}^{n-1}$. We identify $\mathcal{A}$ with a subset of $\tilde{\mathcal{A}}$ by identifying each τ^j with the constant map from $[\lambda_{j-1}, \lambda_j]$ to $\mathbb{R}$. We shall identify $x^j = \mathrm{Im}(\tau^j) \in \mathrm{Im}(\mathbb{H})$ with the corresponding element of $\mathbb{R}^3$ and call it the location of the jth monopole. It follows from the discussion at the end of Section 2 that the hyperkaehler quotient of $\tilde{\mathcal{A}}$ by $\tilde{\mathcal{G}}$ is the same as the hyperkaehler quotient of $\mathcal{A}$ by the $\mathcal{G}$, the subgroup of $\tilde{\mathcal{G}}$ fixing $\mathcal{A}$.

The space $\mathcal{A}$ is a quaternionic vector space and has an inner product induced from $\tilde{\mathcal{A}}$ which is

$$\langle (\tau, a), (\sigma, b) \rangle = \sum_{i=1}^{n} p_i \mathrm{Re}(\tau^i \bar{\sigma}^i) + \sum_{i=1}^{n} \mathrm{Re}(a^i \bar{b}^i), \tag{4.1}$$

where $p_j = \lambda_j - \lambda_{j-1}$.

The group $\mathcal{G}$ is the group of all $g \in \mathcal{G}$ such that each $\mathrm{d}g^j$ is a constant. Such a g can be written as

$$g^j(s) = \exp\left(\frac{2\mathrm{i}\pi}{p_j}((W_+^j - W_-^j)s + W_-^j\lambda_j - W_+^j\lambda_{j-1})\right). \tag{4.2}$$

Notice that $g^j(\lambda_j) = \exp(2\pi \mathrm{i} W_-^j)$ and $g^{j+1}(\lambda_j) = \exp(2\pi \mathrm{i} W_+^i)$ so that condition (3.1) for g to be in $\tilde{\mathcal{G}}$, when applied to a g of the form (4.2) is

$$\begin{aligned} W_-^1 &\in \mathbb{Z}, \\ W_-^2 - W_+^1 &\in \mathbb{Z}, \\ &\vdots \\ W_-^n - W_+^{n-1} &\in \mathbb{Z}, \\ -W_+^n &\in \mathbb{Z}. \end{aligned} \tag{4.3}$$

The numbers W_-^j and W_+^j are not uniquely determined by g. They can be changed by adding to both of them the same integer.

Define a group $\hat{\mathcal{G}}$ to be the group of all $2n$-tuples of real numbers

$$((W_-^1, W_+^1), (W_-^2, W_+^2), \ldots, (W_-^n, W_+^n))$$

satisfying conditions (4.3). Define $\mathcal{G}$ to be the quotient of $\hat{\mathcal{G}}$ by the subgroup consisting of all $2n$-tuples of the form $((k_1, k_1), \ldots, (k_n, k_n))$ where each of the k_i is an integer. We identify $\mathcal{G}$ with its image in the group $\tilde{\mathcal{G}}$ by the map in (4.2). The group $\mathcal{G}$ acts on $\mathcal{A}$ by

$$g(\tau, a) = (g\tau, ga),$$

where

$$\begin{aligned} (g\tau)^j &= \tau^j + \frac{2\pi}{p_j}(W_+^j - W_-^j), \\ (ga)^j &= a^j \exp(2\pi \mathrm{i} W_+^j) = a^j \exp(2\pi \mathrm{i} W_-^{j+1}). \end{aligned} \tag{4.4}$$

Since $\hat{\mathcal{G}} \to \mathcal{G}$ is a discrete quotient the Lie algebra of $\mathcal{G}$ is the Lie algebra of $\hat{\mathcal{G}}$ and hence consists of all $2n$-tuples of the form

$$((1, w^1), (w^1, w^2), \ldots, (w^{n-2}, w^{n-1}), (w^{n-1}, 1)). \tag{4.5}$$

We identify an element such as (4.5) with the corresponding $n-1$ tuple $w = (w^1, \ldots, w^{n-1})$ and denote both by w. The vector field $\iota(w)$ on $\mathcal{A}$ generated by w is

$$\iota(w)(\tau, a) = ((\dot{\tau}^1, \ldots, \dot{\tau}^n), (\dot{a}^1, \ldots, \dot{a}^{n-1}))$$

where

$$\dot{\tau}^1 = \frac{2\pi}{p_1}(w^1),$$
$$\dot{\tau}^2 = \frac{2\pi}{p_2}(w^1 - w^2),$$
$$\vdots$$
$$\dot{\tau}^{n-1} = \frac{2\pi}{p_{n-1}}(w^{n-2} - w^{n-1}),$$
$$\dot{\tau}^n = \frac{2\pi}{p_n}(-w^{n-1})$$

and

$$\dot{a}^j = 2\pi a^j \mathrm{i} w^j.$$

The moment map μ for the action of $\mathcal{G}$ on $\mathcal{A}$ can be calculated from (2.1) but it is the restriction of that for $\tilde{\mathcal{G}}$ on $\tilde{\mathcal{A}}$ and hence we deduce that $(\tau, a) \in \mu^{-1}(0)$ if and only if

$$\mathrm{Im}(\tau^{j+1}) - \mathrm{Im}(\tau^j) = \tfrac{1}{2} a^j \mathrm{i} \bar{a}^j$$

for each $j = 1, \ldots, n-1$.

Let $\mathcal{A}_0 = \mu^{-1}(0)$ so the moduli space of Nahm data is $\mathcal{N} = \mathcal{A}_0/\mathcal{G}$.

5. The metric on monopoles

By the Nahm transform [13,19] the space $\mathcal{N}$ is diffeomorphic to the space of monopoles of type $(1, 1, \ldots, 1)$. The monopole corresponding to the orbit of (τ, a) can be interpreted as a collection of n particles, located at each of the points $x^j = \mathrm{Im}(\tau^j)$ with phases $\exp(\mathrm{i} p_j \mathrm{Re}(\tau^j))$. Following [14] we define the *centre* of τ by

$$\tau_c = \frac{1}{p} \sum_{i=1}^{n} p_i \tau^i,$$

where

$$p = \sum_{i=1}^{n} p_i.$$

The centre of the monopole is then

$$x_c = \frac{1}{p} \sum_{i=1}^{n} p_i x^i.$$

We define the space of *centred monopoles*, $\mathcal{A}_{0,c}$, to be the subset of $\mathcal{A}_0$ consisting of those (τ, a) with $\tau_c = 0$. Define also

$$\mathcal{G}_c = \left\{ g \mid \sum_{j=1}^{n} W_+^j - W_-^j = 0 \right\}.$$

This is the subgroup of $\mathcal{G}$ which fixes $\mathcal{A}_{0,c}$. We define $\mathcal{N}_c = \mathcal{A}_{0,c}/\mathcal{G}_c$.

We want to define an isomorphism

$$\mathcal{A}_0 \to \mathcal{A}_{0,c} \times \mathbb{H}. \tag{5.1}$$

To construct the isomorphism we first define for any $\tau \in \mathbb{H}$ the element $\hat{\tau} \in \mathbb{H}^n$ by $\hat{\tau} = (\tau, \tau, \dots, \tau)$. Notice that $\hat{\tau}_c = \tau$. So given a monopole $(\tau, a) \in \mathcal{A}_0$ we can centre it by defining $(\tau - \hat{\tau}_c, a) \in \mathcal{A}_{0,c}$. The map in (5.1) is then defined to send (τ, a) to the pair $((\tau - \tau_c, a), \tau_c)$ consisting of the corresponding centred monopole and the centre of the monopole. This map has inverse given by $((\tau, a), \sigma) \mapsto (\tau + \hat{\sigma}, a)$.

Notice that the isomorphism in (5.1) commutes with the action of $\mathcal{G}_c$ on both spaces. Define $\hat{\mathcal{N}} = \mathcal{A}_0/\mathcal{G}_c$ and $\mathcal{N}_c = \mathcal{A}_{0,c}/\mathcal{G}_c$. The map in (5.1) also preserves inner products if we give $\mathbb{H}$ the standard metric multiplied by a factor of p. Hence, we have an isometry

$$\hat{\mathcal{N}} \to \mathcal{N}_c \times \mathbb{H},$$

where $\mathbb{H}$ has the usual inner product multiplied by a factor of p. Finally, notice that

$$\sum_{i=1}^{n} W_+^i - W_-^i = -W_-^1 + (W_+^1 - W_-^2) + \dots + W_+^n$$

is an integer by (4.3). So, $\mathcal{G}_c$ is the kernel of a surjective homomorphism $\mathcal{G} \to \mathbb{Z}$ that sends g to $\sum_{i=1}^n W_+^i - W_-^i$ and hence $\mathcal{G}/\mathcal{G}_c$ is isomorphic to $\mathbb{Z}$. Noting, from (4.4), that the action of $\mathcal{G}_c$ leaves the imaginary part of τ alone we conclude that

$$\mathcal{N} = \mathbb{R}^3 \times \frac{\mathcal{N}_c \times \mathbb{R}}{\mathbb{Z}}.$$

We now construct the metric on $\mathcal{N}_c$.

6. The metric on centred monopoles

If (τ, a) is in $\mathcal{A}_{0,c}$ then the vector $x = \mathrm{Im}(\tau)$ is determined by the equations $x^{j+1} - x^j = (\frac{1}{2}) a^j i \bar{a}^j$. So, (τ, a) is determined by the pair $(\mathrm{Re}(\tau), a)$. It is straightforward to show that the orbit of (τ, a) under $\mathcal{G}_c$ contains exactly one pair of the form (τ', a') with $\mathrm{Re}(\tau') = 0$. It follows that $\mathcal{N}_c$ has the topology of $\mathbb{H}^{n-1}$.

In the case that $n = 2$ Connell calculated explicitly the hyperkaehler quotient metric on $\mathbb{H}$. In the case at hand that calculation is more involved and it is simpler to use an approach due to Hitchin [6]. The $n-1$ dimensional torus $T^{n-1} = U(1)^{n-1}$ acts on the space $\mathcal{N}_c$ preserving the hyperkaehler metric by rotating each of the a^i. The moment map for the ith of these actions is given by $\mu_i(\tau, a) = 2\pi\, \mathrm{Im}(\tau^i - \tau^{i+1})$ for each $i = 1, \dots, n-1$. This action is free if none of the a^j vanish. Let $\mathcal{A}'_{0,c}$ be the set of (τ, a) such that none of the a^j vanish. Denote by M the image of $\mathcal{A}'_{0,c}$ in $(\mathbb{R}^3)^{n-1}$ under the moment map. The moment map $\mathcal{A}'_{0,c} \to M$ realises $\mathcal{A}'_{0,c}$ as a T^{n-1} bundle over M. The inner product on $\mathcal{A}'_{0,c}$ allows us to define a horizontal subspace orthogonal to the T^{n-1} action at each point of $\mathcal{A}'_{0,c}$ and hence we can define a connection on $\mathcal{A}'_{0,c} \to M$. This defines a one-form $\alpha = (\alpha^1, \dots, \alpha^{n-1})$ corresponding to projecting onto the vertical subspace. By generalising the calculation of Hitchin [6, Section IV.4] it is possible to show that the metric on $\mathcal{A}'_{0,c}/\mathcal{G}_c$ must have the form

$$g = \sum_{i,j=1}^{n-1} K_{ij}^{-1} \sum_{a=1}^{3} \mathrm{d}\mu_i^a \, \mathrm{d}\mu_j^a + \sum_{i,j=1}^{n-1} K_{ij}\alpha^i\alpha^j \tag{6.1}$$

for some matrix valued function K_{ij} which is constant in the torus directions. If η_i are the generators of the torus action then Hitchin's result gives

$$K_{ij} = g(\eta_i, \eta_j).$$

We now wish to calculate the K_{ij}.

To calculate the η_i we have to split them into a vector $\iota(\xi_i)$ in the direction of the $\mathcal{G}_c$ action and an orthogonal vector $\hat{\eta}_i = \eta_i - \iota(\xi_i)$. Then we have that

$$K_{ij} = g(\eta_i, \eta_j) = \langle \hat{\eta}_i, \hat{\eta}_j \rangle,$$

where $\langle\, ,\rangle$ is the inner product defined in (4.1). Using the orthogonality we deduce that

$$K_{ij} = \langle \eta_i, \eta_j \rangle - \langle \iota(\xi_i), \eta_j \rangle.$$

The condition that defines the $\iota(\xi_i)$ is the requirement that $\eta_i - \iota(\xi_i)$ be horizontal, that is

$$\langle \eta_i - \iota(\xi_i), \iota(\rho) \rangle = 0$$

for all $\rho \in L\mathcal{G}_C$. Expanding this we have that

$$\langle \eta_i, \iota(\rho) \rangle = \langle \iota(\xi_i), \iota(\rho) \rangle.$$

The vector η_l is

$$\eta_l(\tau, a) = (0, (0, \dots, 2\pi \mathrm{i} a^l, \dots, 0))$$

and hence we have

$$\langle \eta_l, \iota(\rho) \rangle = 4\pi^2 \rho_l |a^l|^2.$$

The other inner product is

$$\begin{aligned}\langle \iota(\xi_l), \iota(\rho) \rangle &= 4\pi^2 \left(\sum_{k=1}^{n} \frac{1}{p_k}(\xi_l^{k-1} - \xi_l^k)(\rho^{k-1} - \rho^k) \right) + 4\pi^2 \sum_{k=1}^{n-1} |a^k|^2 \xi_l^k \rho^k \\ &= 4\pi^2 \left(\sum_{k=1}^{n} -\rho^k\Big(\frac{1}{p_k}(\xi_l^{k-1} - \xi_l^k) + \frac{1}{p_{k+1}}(\xi_l^k - \xi_l^{k+1}) + |a^k|^2 \xi_l^k \right).\end{aligned} \tag{6.2}$$

If we equate each coefficient of ρ^k in (6.2) to zero we can put the defining condition for $\iota(\xi_i)$ into the following matrix form. We let $\xi = (\xi_l^k)$ be a matrix with rows labelled by l

and columns labelled by k. We denote by X the diagonal matrix whose lth diagonal entry is $|a^l|^2$. Finally we denote by P the following matrix:

$$\mathrm{P} = \begin{pmatrix} 1/p_1 + 1/p_2 & -1/p_2 & 0 & \dots & 0 & 0 \\ -1/p_2 & 1/p_2 + 1/p_3 & -1/p_3 & \dots & 0 & 0 \\ 0 & -1/p_3 & 1/p_3 + 1/p_4 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1/p_{n-2} + 1/p_{n-1} & -1/p_{n-1} \\ 0 & 0 & 0 & \dots & -1/p_{n-1} & 1/p_{n-1} + 1/p_n \end{pmatrix}.$$

Then the condition satisfied by ξ becomes the matrix equation

$$\xi(P + X) = X$$

and the matrix we are trying to find, K, satisfies

$$K = 4\pi^2(1 - \xi)X.$$

It follows that

$$K^{-1} = \frac{1}{4\pi^2}(P^{-1} + X^{-1}).$$

We conclude that the metric on $\mathcal{N}_c$ is of the form

$$g = \sum_{l,j=1}^{n-1} (P^{-1} + X^{-1})_{lj} \sum_{a=1}^{3} \mathrm{d}y_l^a \, \mathrm{d}y_j^a + 4\pi^2 \sum_{l,j=1}^{n-1} (P^{-1} + X^{-1})_{lj}^{-1} \alpha^l \alpha^j, \tag{6.3}$$

where $y_l = x^{l+1} - x^l = (1/2\pi)\mu^l$.

To finish we want to compare our result (6.3) to formula (7.5) in [15]. Except for rescalings the only question is to show that their matrix μ_{ij} is the matrix P_{ij}^{-1}. To do this we have to calculate μ_{ij} in the manner they suggest. We reintroduce the centre of mass co-ordinate x_C. This means we replace P^{-1} in (6.3) by $\hat{P}^{-1}$ where

$$\hat{P}^{-1} = \begin{pmatrix} p & 0 \\ 0 & P^{-1} \end{pmatrix}.$$

Then we consider the effect of the co-ordinate change from the co-ordinates x^i to the co-ordinates (x_C, y^i). This is the result of applying the linear transformation

$$Z = \begin{pmatrix} p_1/p & p_2/p & p_3/p & \cdots & p_{n-1}/p & p_n/p \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

to the co-ordinates x^i. Hence the matrix of the metric in terms of the co-ordinates x^i is given by $Z^t \hat{P}^{-1} Z$. Let D denote the diagonal matrix with entries $p_1, \dots, p_n$. It is straightforward to check that $ZD^{-1}Z^t = \hat{P}$ and hence $Z^t \hat{P}^{-1} Z = D$. This shows that $Z^t \hat{P}^{-1} Z$ agrees with the constant term in M_{ii} in [15] (their m_i is our p_i). So, we conclude that the μ_{ij} in [15] is indeed P_{ij}^{-1}. The metric on $\mathcal{N}_c$ is therefore the same asymptotically as the metric on the monopole moduli space calculated in [15].

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